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Influence of viscous dissipation on Bénard convection

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The approximations implicit in Bénard convection have been modified to include viscous dissipation. It is shown that both the influence of an adiabatic temperature gradient and of viscous dissipation are governed by the same dimensionless parameter $Di = \alpha gh/c_p$. Numerical calculations of finite amplitude convection are given for finite values of Di. It is found that increasing Di decreases flow velocities and finally stabilizes the flow.

1. Introduction

In forced convection, viscous dissipation is not significant unless the Mach number is of order one. However, in natural convection viscous dissipation may be important if the body force is large or if the length scale of the problem is large. Ostrach (1952, 1958) has shown that viscous dissipation plays an important role in natural convection in vertical channels and Gebhart (1962, 1971) has studied the role of viscous dissipation in the flow on vertical heated plates. These authors found that the non-dimensional parameter $Di = \alpha gh/c_p$ (α = coefficient of thermal expansion, g = acceleration due to gravity, h = length scale of the problem, c_p = specific heat at constant pressure) determined the influence of viscous dissipation, and it has been called the dissipation number.

In this paper the approximations implicit in the Bénard problem have been modified to include the role of viscous dissipation. It should be noted that the roles of viscous dissipation and compressibility are of the same order when Gruneisen's constant is of order one. Since it is known empirically that Gruneisen's constant is of order one for liquids and gases, the effects of viscous dissipation and compressibility should be considered together for real substances. In this paper only the effects of viscous dissipation are considered. The object is to understand better the role of viscous dissipation in natural convection, in particular the role of the parameter Di. Numerical computations of finite amplitude convection will be presented; it will be shown that viscous dissipation is directly coupled with the adiabatic temperature gradient.

2. Formulation of the problem

We shall consider a layer of fluid which is confined between two horizontal boundaries and is heated from below. The applicable equations for steady thermal convection in a laminar, Newtonian fluid are (Batchelor 1967, p. 164)

$$\partial(\rho u_i)/\partial x_i = 0, \tag{1}$$

$$\rho u_i \frac{\partial u_k}{\partial x_i} = -\frac{\partial p}{\partial x_k} + \frac{\partial \tau_{ik}}{\partial x_i} + \rho g \delta_{k3}, \tag{2}$$

$$\rho u_i c_p \frac{\partial T}{\partial x_i} - T\alpha u_i \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + \tau_{ij}\frac{\partial u_i}{\partial x_j}, \tag{3}$$

$$\tau_{ij} = \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \tfrac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) + \tfrac{1}{3}\zeta\delta_{ij}\frac{\partial u_k}{\partial x_k}, \tag{4}$$

where $(0, 0, 1)$ is the unit vector in the direction of gravity, k the thermal conductivity, η the viscosity, δ_{ij} the Kronecker delta ($\delta_{ij} = 1$ when $i = j$; $\delta_{ij} = 0$ when $i \neq j$) and ζ is the second or bulk viscosity. The generalized linear equation of state is

$$\rho = \rho_0[1 - \alpha(T - T_1) + \chi p], \tag{5}$$

where χ is the isothermal compressibility and ρ_0 and T_1 are the density and temperature at the upper boundary of the fluid layer.

Clearly it is desirable to make as many approximations as possible in the above equations. The approach usually taken is to set $\chi = 0$ and assume that the density is constant except in the body-force term of the momentum equation. This is the Boussinesq approximation and it has been partially justified by Mihaljan (1962). Applying this approximation to (1)-(5) along with the assumption of constant fluid properties and the condition

$$p = \rho_0 g z + P \tag{6}$$

yields

$$\partial u_i/\partial x_i = 0, \tag{7}$$

$$\rho_0 u_i \frac{\partial u_k}{\partial x_i} = -\frac{\partial P}{\partial x_k} + \eta\frac{\partial^2 u_k}{\partial x_i^2} - \rho_0 \alpha g (T - T_1)\delta_{k3}, \tag{8}$$

$$\rho_0 u_i c_p \frac{\partial T}{\partial x_i} - T\alpha\rho_0 g u_z = k\frac{\partial^2 T}{\partial x_i^2} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j}. \tag{9}$$

The second term on the left side of (9) represents the adiabatic temperature gradient considered by Jeffreys (1930) for the linear stability problem.

We next introduce the non-dimensional variables

$$\bar{x}_i = \frac{x_i}{h}, \quad \bar{u}_i = \frac{u_i h}{\kappa}, \quad \bar{P} = \frac{Ph^2}{\rho_0 \nu\kappa}, \quad \bar{T} = \frac{T - T_1}{T_2 - T_1},$$

and non-dimensional parameters

$$Ra = \frac{\alpha g (T_2 - T_1) h^3}{\nu\kappa}, \quad Pr = \frac{\nu}{\kappa}, \quad \theta = \frac{T_1}{T_2 - T_1}$$

($\nu = \eta/\rho_0$ and $\kappa = k/\rho_0 c_p$) in addition to Di, which already has been defined; T_2 is the temperature of the lower boundary. The parameter Ra is the Rayleigh number and Pr the Prandtl number. Substitution of the non-dimensional variables and parameters into (7)–(9) gives

$$\partial \bar{u}_i/\partial \bar{x}_i = 0, \tag{10}$$

$$\frac{1}{Pr}\bar{u}_i \frac{\partial \bar{u}_k}{\partial \bar{x}_i} = -\frac{\partial \bar{P}}{\partial \bar{x}_k} + \frac{\partial^2 \bar{u}_k}{\partial \bar{x}_i^2} - Ra\,\bar{T}, \tag{11}$$

$$\bar{u}_i \frac{\partial \bar{T}}{\partial \bar{x}_i} - Di(\bar{T}+\theta)\,\bar{u}_z = \frac{\partial^2 \bar{T}}{\partial \bar{x}_i^2} + \frac{Di}{Ra}\left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j} + \frac{\partial \bar{u}_j}{\partial \bar{x}_i}\right)\frac{\partial \bar{u}_j}{\partial \bar{x}_i}. \tag{12}$$

It is of interest to consider the volume integral of (12) over a convection cell. The convection terms and the conduction terms do not contribute to such an integral. However, the viscous dissipation term is positive definite and represents a source of heat; therefore its integral is a positive quantity. The integral of the viscous dissipation term must be balanced by the volume integral of the term $Di(\bar{T}+\theta)\,\bar{u}_z$, which represents the adiabatic temperature gradient. Since the effects of viscous dissipation and adiabatic temperature gradient must cancel, it is necessary that they both depend upon the same dimensionless parameter Di.

3. Finite amplitude convection

In order to investigate the influence of the parameter Di on finite amplitude Bénard convection numerical solutions of (10)–(12) have been obtained for two-dimensional cellular convection in a fluid layer. Periodic flow with a wavelength twice the height of the layer was assumed, furthermore mirror symmetry about each half-wavelength was assumed. It was also assumed that the Prandtl number is large so that the inertial terms of the momentum equation could be neglected. A finite difference technique was used; central space differences were used except for the convection terms in the energy equation. For these a special three-point non-central difference method described by Torrance (1968) was used. Rigid isothermal boundary conditions were applied. Assuming an initial flow and temperature distribution the difference equations were solved for the final steady-state flow by iterative extrapolation. With the above assumptions the calculations were carried out using a square 10×10 grid. One case was checked using a 20×20 grid; the streamlines and isotherms were virtually identical to the corresponding 10×10 grid calculation. The ability of this method to refine flow patterns with relatively coarse grids was previously demonstrated by Torrance & Turcotte (1971) and by Hsui, Turcotte & Torrance (1972); it is estimated that calculated temperatures and velocities are accurate within 5 %.

Streamlines and isotherms for $\theta = 0$, $Ra = 10^4$ and $Di = 0$, 0·5, 1 and 1·5 are shown in figure 1. As indicated by the decrease in $\psi_{\max}$ the velocities decrease as Di is increased. For finite Di the flow is no longer symmetric since there is internal heat production. The results for $\theta = 0$, $Ra = 10^5$ and $Di = 0$, 1 and 2

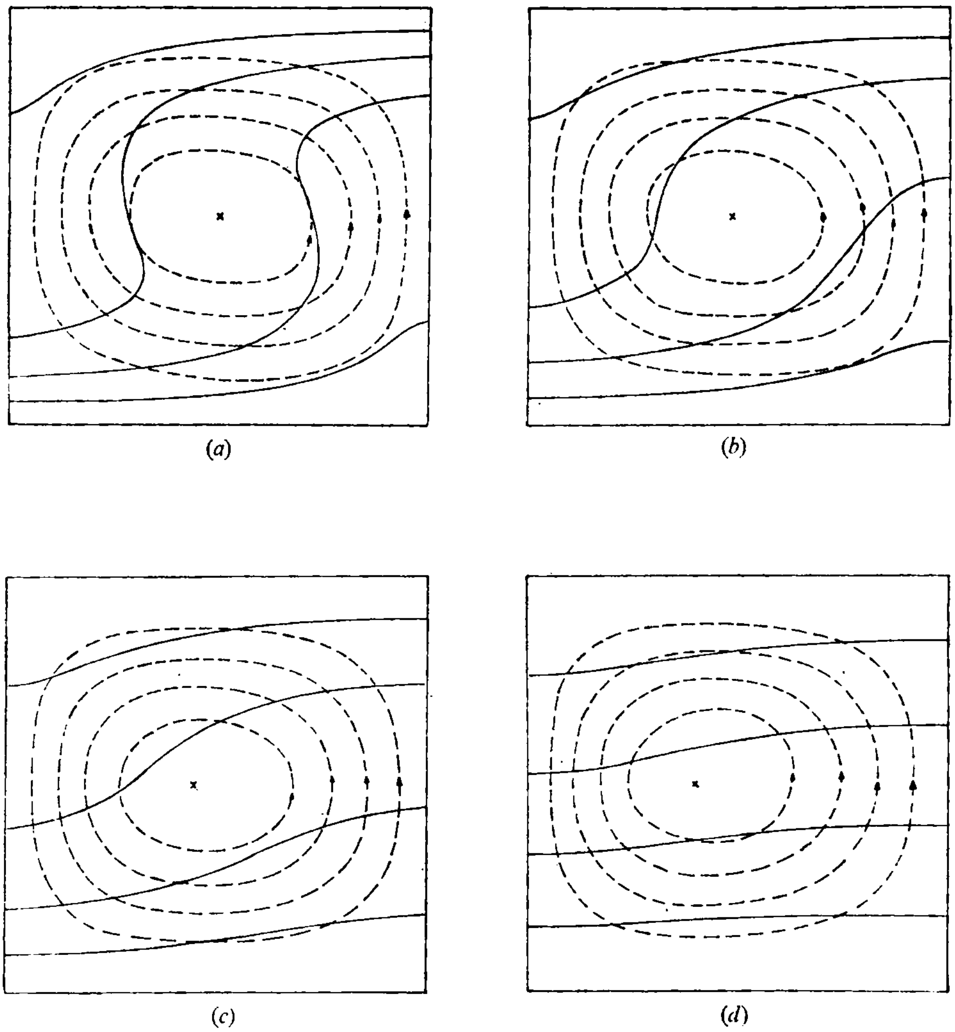

FIGURE 1. Streamlines (dashed lines) and isotherms (solid lines) for $Ra = 10^4$, $\theta = 0$ and various values of Di. (a) $Di = 0$, $\psi_{max} = 9{\cdot}0612$. (b) $Di = 0{\cdot}5$, $\psi_{max} = 7{\cdot}1227$. (c) $Di = 1$, $\psi_{max} = 4{\cdot}9487$. (d) $Di = 1{\cdot}5$, $\psi_{max} = 1{\cdot}9392$.

are given in figure 2. Again the velocities decrease with increasing Di. Also the flow breaks into two cells when $Di = 2$. The decrease in convection with increasing Di can be attributed to the increase in the adiabatic temperature gradient. Since it is the difference between the local temperature gradient and the local adiabatic temperature gradient that drives convection, an increase in the adiabatic temperature gradient (increase in Di) with a fixed applied temperature difference (fixed Ra) reduces convection. Since the adiabatic temperature gradient is proportional to temperature it is a maximum in the lower part of the layer, particularly for the case $\theta = 0$ considered here. This is the reason why for $Di = 2$ and $Ra = 10^5$ the lower part of the layer is stabilized, convection is restricted to the upper part of the layer and its wavelength decreases as shown

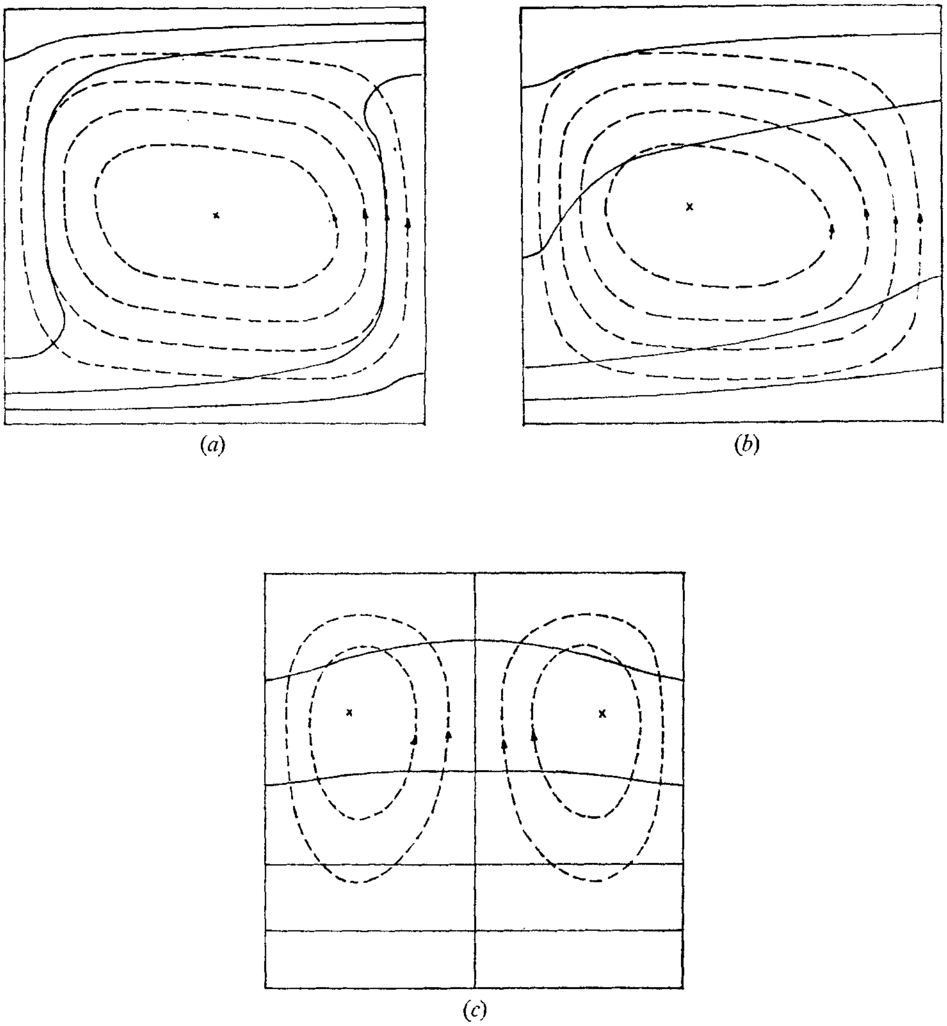

FIGURE 2. Streamlines (dashed lines) and isotherms (solid lines) for $Ra = 10^5$, $\theta = 0$ and various values of Di. (*a*) $Di = 0$, $\psi_{max} = 37{\cdot}028$. (*b*) $Di = 1$, $\psi_{max} = 21{\cdot}821$. (*c*) $Di = 2$, $\psi_{max} = 4{\cdot}067$.

in figure 2. The heating associated with viscous dissipation does not enhance convection either locally or throughout the cell.

To illustrate the decrease in heat transfer with increasing Di the Nusselt number Nu for each calculation is shown in figure 3. The Nusselt numbers decrease with increasing Di towards the conduction value of unity. The values of Di corresponding to the onset of convection for Rayleigh numbers of 10^4 and 10^5 are in good agreement with the finite amplitude calculations. These values were obtained from the appropriate stability analysis. The decrease in convective heat transfer associated with the increase in dissipation is directly coupled to the increased stability associated with the increasing adiabatic temperature gradient.

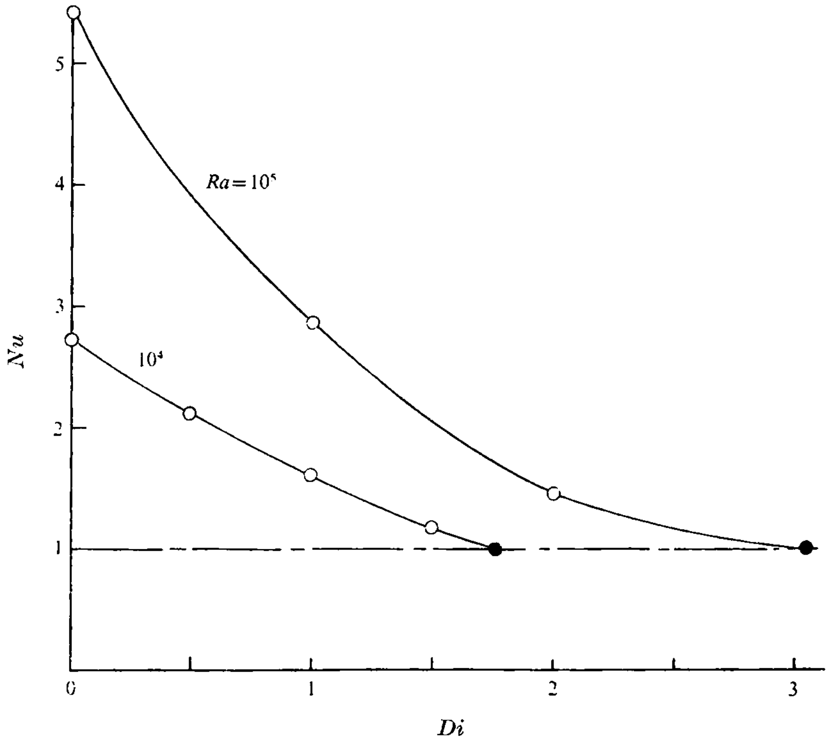

FIGURE 3. Dependence of the Nusselt number on the parameter Di for Rayleigh numbers of 10^4 and 10^5 with $\theta = 0$. ○, numerical calculations; ●, stability analysis.

Viscous dissipation significantly influences Bénard convection when the parameter Di is of order one. We have shown that viscous dissipation and the adiabatic temperature gradient must be considered simultaneously for finite amplitude Bénard convection. The principal effect of increasing Di at a fixed Rayleigh number is to reduce thermal convection.

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